The density of the real parts of the zeros of the entire functions $\{1 + 2^z + \cdots + n^z : n \geq 2\}$

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Abstract

Our purpose in this paper is to study the behavior of the real parts of the zeros of the functions

$$G_n(z) = 1 + 2^z + \cdots + n^z,\ n \geq 2.$$

Firstly, we will consider some particular values of n for which the real parts of the zeros of $G_n(z)$ are dense in some intervals of the real line. Secondly, by denoting

$$S_n = \{\lambda < 0 : \exists \beta \in \mathbb{R} \text{ such that } G_n(\lambda + \beta i) = 0\},$$

and $\overline{S}_n = \mathrm{cl}(S_n)$ the closure of S_n, we will establish some conditions for which we can assure $\overline{S}_{n-1} \subset \overline{S}_n$, for all $n \in \mathbb{N}$, $n \geq 2$.

1. Introduction

We show the following results (see [2], [3])

- We consider exponential polynomials of the form

$$P_n(z) = a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z},\ \text{with } a_j \in \mathbb{C},\ \lambda_j \in \mathbb{R},\ \forall j : 1 \leq j \leq n.$$

- All the zeros of $P_n(z)$ are situated in some strip parallel to the real axis
- $G_n(z) = 1 + 2^z + \cdots + n^z$ is an entire function of order 1 for each fixed integer $n \geq 2$
- $G_n(z)$ is a function of exponential type $\sigma = \ln n$
- The sequence $\{G_n(z) : n \geq 2\}$ approaches the Riemann zeta function for $\mathrm{Re}\, z < -1$
- $G_n(z)$ has infinitely many zeros for each fixed integer $n \geq 2$
- The functions $G_n(z)$ do not have all the zeros on the imaginary axis, except for $n = 2$

2. Preliminaries

Theorem 1 (Equivalent to theorem 3.1 of [1]) *Let $P_n(z) = \sum_{j=1}^n a_j e^{b_j z}$ be an exponential polynomial with $a_j \in \mathbb{C}$ and $b_j \in \mathbb{R}\ \forall j = 1, \ldots, n$. Let $B = \{b_{j_k}\}_{k=1}^m$ be the base of the $\mathbb{Q}-$vectorial space generated by $\{b_j\}_{j=1}^n$. Let $b = (b_{j_1}, b_{j_2}, \ldots, b_{j_m})$ and $c_j = (c_j^1, \ldots, c_j^m)$, with $c_j^k \in \mathbb{Q}$, such that $b_j =< c_j, b >$, where $< \cdot, \cdot >$ denotes the scalar product.*

We define the function

$$\begin{aligned} F_{n,m} : \mathbb{R} \times [0, 2\pi)^m &\to \mathbb{C} \\ (t, x_1, \ldots, x_m) &\mapsto \sum_{j=1}^n |a_j| e^{b_j t} e^{<c_j,(x_1,\ldots,x_m)>i}. \end{aligned}$$

If there exists an interval $[a, b] \subset \mathbb{R}$ such that $\forall t \in [a, b]$ we can find a vector $(x_1', \ldots, x_m') \in [0, 2\pi)^m$ with $F_{n,m}(t, x_1', \ldots, x_m') = 0$, then the projection of the zeros of $P_n(z)$ in the real line is dense in $[a, b]$. The converse is also true.

3. Density for $n = 4$

We consider the entire function $G_4(z) = 1 + 2^z + 3^z + 4^z$. Let

$$\begin{aligned} F_{4,2} : \mathbb{R} \times [0, 2\pi)^2 &\to \mathbb{C} \\ (t, x_1, x_2) &\mapsto 1 + 2^t e^{x_1 i} + 3^t e^{x_2 i} + 4^t e^{2x_1 i} \end{aligned}$$

be the function that appears in the Theorem 1 and

$$\begin{aligned} f_1 : \mathbb{R} \times [0, 2\pi) &\to \mathbb{C} \\ (t, x_1) &\mapsto 1 + 2^t e^{x_1 i} + 4^t e^{2x_1 i}. \end{aligned}$$

Then, the distance function, $d(\cdot, 0)$ verifies that $d(f_1(t, 0), 0) = 1 + 2^t + 4^t > 3^t$ and, taking $w = 2^t$, the distance $d(f_1(t, \frac{3\pi}{4}), 0)$ satisfies

$$\left(\frac{w}{\sqrt{2}} - w^2\right)^2 + \left(1 - \frac{w}{\sqrt{2}}\right)^2 - w^3 < 0 \Rightarrow d\left(f_1\left(t, \frac{3\pi}{4}\right), 0\right) - 3^t < 0$$

and the last polynomial have two real zeros, $w \approx \pm 0.75$, that is, $t \approx -0.41$.

Therefore

$$d\left(f_1\left(t, \frac{3\pi}{4}\right), 0\right) - 3^t < 0 \ \text{ if } t > -0.41.$$

Finally, using the continuity of the distance function, for each $t > -0.41$, there exists $x_1^t \in [0, 2\pi)$ such that

$$d\left(f_1\left(t, x_1^t\right), 0\right) - 3^t = 0,$$

that is

$$f_1\left(t, x_1^t\right) = 3^t e^{iA_t} \ \text{ for some } A_t \in [0, 2\pi)$$

and consequently

$$F_{4,2}(t, x_1^t, A_t + \pi) = 0.$$

Now, taking into account the Theorem 1, we can assure the density of the projection of the zeros of $G_4(z)$ if $t > -0.41$.

4. From $n - 1$ to n, if n is prime

We denote by R_n the subset of the real numbers determined by

$$R_n = \{\lambda < 1 : \exists \beta \in \mathbb{R} \text{ such that } G_n(\lambda + \beta i) = 0\},$$

and $\overline{R}_n = \mathrm{cl}(R_n)$ the closure of R_n.

Theorem 2 *Let n be a prime number greater than 3, then*

$$\overline{R}_{n-1} \subset \overline{R}_n.$$

5. From $n - 1$ to n, if n is not prime

We denote by S_n the subset of the real numbers determined by

$$S_n = \{\lambda < 0 : \exists \beta \in \mathbb{R} \text{ such that } G_n(\lambda + \beta i) = 0\},$$

and $\overline{S}_n = \mathrm{cl}(S_n)$ the closure of S_n.

Theorem 3 *Let n be an integer number greater than 3, n not prime, and let $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in [0, 2\pi)^k$ be the vector whose existence is assured by the theorem 1. Then, if $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ verifies $\cos\left(\alpha_k - < c_j \cdot (\alpha_1, \ldots, \alpha_{k-1}) >\right) > \frac{1}{2}$, we have*

$$\overline{S}_{n-1} \subset \overline{S}_n,$$

6. Relation between several sets

Let f be a positive real function, we define the function $E_n(z) = n^z$ and the set

$$S_n^{(f)} := \{a \in \mathbb{R}, \exists b \in \mathbb{R} : |G_n(a + bi)| \leq f(a)\}.$$

Proposition 1 *If n is prime, then*

$$S_n = S_{n-1}^{(E_n)} \supset S_{n-1}^{\left(n^{\inf S_{n-1}}\right)}.$$

Proposition 2 *If n is not prime, then*

$$S_{n-1} \subseteq S_n^{(E_n)}.$$

Proposition 3 *For all $n \geq 2$,*

$$S_n \subseteq S_{n-1}^{(E_n)}.$$

References

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